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ORIGINAL PAPER

A conservation law for a generalized chemical Fisher equation

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Abstract In this work we study a generalized Fisher equation with variable coefficients from the point of view of the theory of symmetry reductions in partial differential equations. There is a widespread occurrence of nonlinear phenomena in physics, chemistry and biology. This clearly necessitates a study of conservation laws in depth and of the modeling and analysis involved. We determine the class of these equations which are nonlinearly self-adjoint. By using a general theorem on conservation laws proved by Nail Ibragimov and the symmetry generators we find some conservation laws for some of these partial differential equations without classical Lagrangians.

Keywords Fisher equation · Lie symmetries · Self-adjointness · Weak self adjointness · Nonlinear

1 Introduction

The analysis and study of the Fisher equation is used to model heat and reaction-diffusion problems applied to mathematical biology, physics, astrophysics, chemistry, genetics, bacterial growth problems as well as development and growth of solid tumours. It is well known that nonlinear reaction-diffusion equations play an important role in dissipative dynamical systems. Several examples are provided through modeling of certain phenomena in chemical engineering.

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When reaction kinetics and diffusion are coupled, traveling waves of chemical concentration can effect a biochemical change much faster than straight diffusional processes. This usually gives rise to reaction-diffusion equations which in one dimensional space can look like

$$u_t = Du_{xx} + f(u)$$

for a chemical concentration u, where D is the diffusion coefficient, and $f(u)$ represents the kinetics.

Applications of nonlinear reaction-diffusion models can be seen from nucleation kinetics, neutron action in the reactor, biological invasion in contemporary ecology, wave propagation in nerve fibers, the smallest scales of theortical particle physics and so on [1].

When the chemical reaction

$$A + B \leftrightarrow 2A$$

takes place in a setting, where the density of species B can be assumed as constant and species A is subject to one-dimensional diffusion, then the dynamics of the density $u(x, t)$ of species A can be described (after non-dimensionalization) by the KPP-Fisher equation

$$u_t = Du_{xx} + u(c - u)$$

with the diffusion coefficient $D > 0$. This equation has two constant equilibrium states, $u \equiv 0$ and $u \equiv c > 0$, the former linearly unstable and the latter linearly stable.

In a previous paper [2] we have considered the generalized Fisher equation

$$u_t = f(u) + (g(u)u_x)_x \,, \tag{1}$$

whose g is the diffusion coefficient depending on the variable u, being x and t the independent variables, and $f(u)$ an arbitrary function. For (1) we have determined the subclasses of equations which are nonlinearly self-adjoint. By using the Lie generators of (1) and the notation and techniques of [3], we got some non-trivial conservation laws for equation (1).

The equation analyzed in this paper is a generalized Fisher equation with variable coefficients, where g is the diffusion coefficient depending on the variable u, being x and t the independent variables, $f(u)$ an arbitrary function and $c(x)$ an arbitrary function depending on the space variable x. Let $u(x, t)$ denote the density of tumor cells

$$u_t = f(u) + \frac{1}{c(x)}\,(c(x)g(u)u_x)_x \,. \tag{2}$$

Equation (2) has been analyzed in different and particular cases by other authors. The transient heat conduction equation with a heat source term following a power law in a rectangular, cylindrical coordinate system has been considered by Moitsheki [4] by using Lie classical symmetries. In Bokhari et al. [5] have considered the following particular case of (2) where $g(u) = u$ and $f(u) = u(1-u)$

$$u_t = u(1-u) + \frac{1}{x}\left[xuu_x\right]_x, \tag{3}$$

for which the authors derived an exact solution in term of the Bessel functions by using Lie classical reductions.

In Bokhari et al. [6] have considered equation (2) but only when $g(u)$ is a linear function $g(u) = \alpha_1 u + \alpha_2$ and $c(x) = x$. They state that a classification of (2) can only be achieved when g is linear in u. When f and g follow a power law they give an stationary solution.

In a different paper (submitted for publication) we have considered equation (2) from the point of view of the theory of symmetry reductions in partial differential equations and we have obtained a complete group classification.

The idea of a conservation law, or more particularly, of a conserved quantity, has its origin in mechanics and physics. Since a large number of physical theories, including some of the 'laws of nature', are usually expressed as systems of nonlinear differential equations, it follows that conservation laws are useful in both general theory and the analysis of concrete systems [7]. In Anco and Bluman [8] gave a general treatment of a direct conservation law method for partial differential equations expressed in a standard Cauchy–Kovaleskaya form

$$u_t = G(x, u, u_x, u_{xx}, \ldots, u_{nx}).$$

In Kara and Mahomed [9] showed how to construct conservation laws of Euler-Lagrange type equations via Noether type symmetry operators associated with partial Lagrangians. In Ibragimov [3] (see also [10]) a general theorem on conservation laws for arbitrary differential equations which do not require the existence of Lagrangians has been proved. This new theorem is based on the concept of adjoint equations for non-linear equations. There are many equations with physical significance which are not self-adjoint. Therefore one cannot eliminate the nonlocal variables from the conservation laws of these equations. In Ibragimov [11] has generalized the concept of self-adjoint equations by introducing the definition of quasi self-adjoint equations.

It happens that many equations having remarkable symmetry properties, are neither self-adjoint nor quasi self-adjoint.

In Gandarias [12] one of the present authors has generalized the concept of quasi-self-adjoint equations by introducing the concept of weak self-adjoint equations. In Ibragimov [13] has generalized this concept and has introduced the concept of non-linear self-adjointness. By using these two recent developments Freire and Sampaio [14] have determined the nonlinearly self-adjoint class of a generalized fifth order equation and by using Ibragimov theorem [10] the authors have established some local conservation laws. In Johnpillai and Khalique [15] have studied the conservation laws of some special forms of the nonlinear scalar evolution equation, the modified Korteweg-De Vries (mKdV) equation with time dependent variable coefficients of damping and dispersion. The authors use the new conservation theorem (Ibragimov [10]) and the partial Lagrangian approach (Kara and Mahomed [9]).

In [12,16] the concept of quasi self-adjoint equations has been generalized by introducing the definition of weak self-adjoint equation in which substitution $v = h(u)$ can be replaced with a more general substitution where h involves not only the variable u but also the independent variables $h = h(x, t, u)$. In constructing conservation laws, it is only important that v does not vanish identically, because otherwise yields the trivial vector $C^i = 0$. Therefore, we can replace the condition $h'(u) \neq 0$ with the weaker condition $h(u) \neq 0$.

In Ibragimov [13] the concept of quasi self-adjoint equations has been generalized by introducing the definition of nonlinearly self-adjoint equation in which substitution $v = h(u)$ can be replaced with a more general substitution where h involves not only the variable u but also its derivatives as well as the independent variables $v = h(x, t, u, u_t, u_x, ..)$.

The aim of this paper is to determine, for (2) the subclasses of equations which are nonlinearly self-adjoint, and to determine some non-trivial conservation laws by using the Lie generators and the notation and techniques in [3].

2 Nonlinearly self-adjoint equations

Consider an sth-order partial differential equation

$$F(x, u, u_{(1)}, \ldots, u_{(s)}) = 0, \tag{4}$$

with independent variables $x = (x^1, \ldots, x^n)$ and a dependent variable u, where $u_{(1)} = \{u_i\}$, $u_{(2)} = \{u_{ij}\}, \ldots$ denote the sets of the partial derivatives of the first, second, etc. orders, $u_i = \partial u/\partial x^i$, $u_{ij} = \partial^2 u/\partial x^i \partial x^j$. The adjoint equation to (4) is

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = 0, \tag{5}$$

with

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = \frac{\delta(vF)}{\delta u}, \tag{6}$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{s=1}^{\infty}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}}, \tag{7}$$

denotes the variational derivatives (the Euler-Lagrange operator), and v is a new dependent variable. Here

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots,$$

are the total differentiations.

Definition Equation (4) is said to be *self-adjoint* if the equation obtained from the adjoint equation (5) by the substitution $v = u$ is identical to the original equation.

Definition Equation (4) is said to be *quasi self-adjoint* if the equation obtained from the adjoint equation (5) by the substitution $v = h(u)$, with a certain function $h(u)$ such that $h'(u) \neq 0$, is identical to the original equation.

Definition Equation (4) is said to be *weak self-adjoint* if the equation obtained from the adjoint equation (5) by the substitution $v = h(x, u)$, with a certain function $h(x, t, u)$ such that $h_u(x, t, u) \neq 0$ or $h_x(x, t, u) \neq 0$ or $h_t(x, t, u) \neq 0$ is identical to the original equation.

Definition Equation (4) is said to be *nonlinearly self-adjoint* if the equation obtained from the adjoint equation (5) by the substitution $v = h(x, u, u_{(1)}, \ldots)$, with a certain function $h(x, u, u_{(1)}, \ldots)$ such that $h(x, u, u_{(1)}, \ldots) \neq 0$, is identical to the original equation.

2.1 The subclass of nonlinearly self-adjoint equations

Let us single out nonlinearly self-adjoint equations from the equations of the form (2). Equation (6) yields

$$
\begin{aligned}
F^* &= \frac{\delta}{\delta u}\left[v(u_t - f(u) - \frac{1}{c(x)}\,(c(x)g(u)u_x)_x)\right] \\
&= -g\, v_{xx} + \frac{c_x\, g\, v_x}{c} - v_t + \frac{c_{xx}\, g\, v}{c} - \frac{(c_x)^2\, g\, v}{c^2} - f_u\, v.
\end{aligned} \tag{8}
$$

Setting $v = h(x, t, u)$ in (8) we get

$$
\begin{aligned}
&-g\, h_u\, u_{xx} - g\, h_{uu}\, (u_x)^2 - 2\, g\, h_{ux}\, u_x + \frac{c_x\, g\, h_u\, u_x}{c} - h_u\, u_t \\
&-g\, h_{xx} + \frac{c_x\, g\, h_x}{c} - h_t + \frac{c_{xx}\, g\, h}{c} - \frac{(c_x)^2\, g\, h}{c^2} - f_u\, h = 0,
\end{aligned}
$$

which yields:

$$
\begin{aligned}
&F^* - \lambda\left(u_t - f(u) - \frac{1}{c(x)}\,(c(x)g(u)u_x)_x\right) \\
&= g\, u_{xx}\,\lambda + g_u\, (u_x)^2\,\lambda + \frac{c_x\, g\, u_x\,\lambda}{c} - u_t\,\lambda + f\,\lambda - g\, h_u\, u_{xx} \\
&\quad -g\, h_{uu}\, (u_x)^2 - 2\, g\, h_{ux}\, u_x + \frac{c_x\, g\, h_u\, u_x}{c} - h_u\, u_t \\
&\quad -g\, h_{xx} + \frac{c_x\, g\, h_x}{c} - h_t + \frac{c_{xx}\, g\, h}{c} - \frac{(c_x)^2\, g\, h}{c^2} - f_u\, h.
\end{aligned}
$$

Comparing the coefficients for u_t, we obtain $\lambda + h_u = 0$ that $h_u(x, t, u) = 0$, we can state the following:

Equation (2) is neither quasi self-adjoint nor weak self-adjoint, however Eq. (2) is nonlinearly self-adjoint, upon the substitution

$$h = h(x,t)$$

for any functions $f = f(u)$, $g = g(u)$ and $c = c(x)$ with $h = h(x,t)$ verifying the following equation

$$g\,h_{x\,x} - \frac{c_x\,g\,h_x}{c} + h_t - \frac{c_{x\,x}\,g\,h}{c} + \frac{(c_x)^2\,g\,h}{c^2} + f_u h\; = 0. \tag{9}$$

3 Conservation laws: general theorem

We use the following theorem on conservation laws proved in [3].
Any Lie point, Lie–Bäcklund or non-local symmetry

$$X = \xi^i(x,u,u_{(1)},\ldots)\frac{\partial}{\partial x^i} + \eta(x,u,u_{(1)},\ldots)\frac{\partial}{\partial u} \tag{10}$$

of Eq. (4) provides a conservation law $D_i(C^i) = 0$ for the simultaneous system (4), (5). The conserved vector is given by

$$\begin{aligned} C^i &= \xi^i\mathcal{L} + W\left[\frac{\partial\mathcal{L}}{\partial u_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) - \cdots\right] \\ &\quad + D_j(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ij}} - D_k\left(\frac{\partial\mathcal{L}}{\partial u_{ijk}}\right) + \cdots\right] + D_jD_k(W)\left[\frac{\partial\mathcal{L}}{\partial u_{ijk}} - \cdots\right] + \cdots, \end{aligned} \tag{11}$$

where W and $\mathcal{L}$ are defined as follows:

$$W = \eta - \xi^j u_j, \quad \mathcal{L} = v\,F\left(x,u,u_{(1)},\ldots,u_{(s)}\right). \tag{12}$$

The proof is based on the following operator identity:

$$X + D_i(\xi^i) = W\frac{\delta}{\delta u} + D_i\mathcal{N}^i, \tag{13}$$

where X is the operator (10) taken in the prolonged form:

$$\begin{aligned} X &= \xi^i\frac{\partial}{\partial x^i} + \eta\frac{\partial}{\partial u} + \zeta_i\frac{\partial}{\partial u_i} + \zeta_{i_1i_2}\frac{\partial}{\partial u_{i_1i_2}} + \cdots, \\ \zeta_i &= D_i(\eta) - u_jD_i(\xi^j), \quad \zeta_{i_1i_2} = D_{i_2}(\zeta_{i_1}) - u_{ji_1}D_{i_2}(\xi^j), \ \ldots . \end{aligned}$$

Let us apply Theorem 1 to the nonlinearly self-adjoint Eq. (2)

$$u_t = f(u) + \frac{1}{c(x)} \left(c(x)g(u)u_x\right)_x ,$$

provided by the generator

$$X = \frac{\partial}{\partial t}.$$

Here $g = kf_u$, $f(u)$, $c(x)$ arbitrary functions and $h = h(x)$ must satisfy

$$c^2 h_{xx} k - c\, c_x h_x k - c\, c_{xx} h k + (c_x)^2 \, h k + c^2 h = 0.$$

In this case

$$W = -u_t.$$

We get the conservation law

$$D_t(C^1) + D_x(C^2) = 0, \tag{14}$$

where

$$\begin{aligned} C^1 &= -g_u h_x u u_x + \frac{c_x g_u h u u_x}{c} + \frac{g h u}{k} - f h + D_x(B), \\ C^2 &= g g_u h_x u u_{xx} - \frac{c_x g g_u h u u_{xx}}{c} + (g_u)^2 \, h_x u \, (u_x)^2 - \frac{c_x \, (g_u)^2 \, h u \, (u_x)^2}{c} \\ &\quad + \frac{c_x g g_u h_x u u_x}{c} - \frac{(c_x)^2 \, g g_u h u u_x}{c^2} + f g_u h_x u - \frac{c_x f g_u h u}{c} - D_t(B), \end{aligned}$$

with

$$B = \left(g h_x k_1 - \frac{c_x g h k_1}{c} \right) u - g h k_1 u_x.$$

We simplify the conserved vector by transferring the terms of the form $D_x(\ldots)$ from C^1 to C^2 and obtain

$$\begin{aligned} C^1 &= -g_u h_x u u_x + \frac{c_x g_u h u u_x}{c} + \frac{g h u}{k} - f h, \\ C^2 &= g g_u h_x u u_{xx} - \frac{c_x g g_u h u u_{xx},}{c} + (g_u)^2 \, h_x u \, (u_x)^2 - \frac{c_x \, (g_u)^2 \, h u \, (u_x)^2}{c} \\ &\quad + \frac{c_x g g_u h_x u u_x}{c} - \frac{(c_x)^2 \, g g_u h u u_x}{c^2} + f g_u h_x u - \frac{c_x f g_u h u}{c}. \end{aligned}$$

4 Conclusions

Mathematical modeling of physical, chemical and biological systems often leads to nonlinear evolution equations. There is a considerable interest in finding conservation laws for these equations which has no Lagrangian. We have found the subclasses of (2) which are nonlinearly self-adjoint. By using the property of nonlinear self-adjointness of (2) and the general theorem of conservation laws [10], we have constructed some nontrivial conservation laws for this generalized Fisher equation associated with symmetries of the differential equations for $g = kf_u$, $f(u)$ and $c(x)$ arbitrary functions.

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